Symbolic Representation of Translatory Motion in Multivarying-Link Mechanisms

Itzhack Y. Bar-Itzhack*
Technion—Israel Institute of Technology, Haifa, Israel

A symbolic representation of position, velocity, and acceleration of any link with respect to another in multilink mechanisms is presented. This representation does away with cumbersome matrix multiplications. Each link may vary in length as well as in orientation. The representation is based on a symbolic representation of the Denavit-Hartenberg 4×4 matrix and its first and second time derivatives. The new representation can be viewed as an extension of the Piogram symbolic representation of coordinate transformation. As with Piograms, once the user masters the new technique, it becomes an easy and powerful tool for obtaining the analytic expressions describing position, velocity, and acceleration in multilink mechanisms.

Nomenclature

${}'A^a_b$	= 4×4 D-H transformation matrix from coordinate system a to b about $\bar{k}_a$ ($\equiv \bar{k}_b$)
A^a_b	= coordinate transformation matrix from coordinate system a to b about $\bar{k}_a$ ($\equiv \bar{k}_b$)
a, b	= general coordinate system
D^a_b	= coordinate transformation matrix from system a to b
$\bar{i}_a$	= unit vector along x axis of coordinate system a
$\bar{j}_a$	= unit vector along y axis of coordinate system a
k	= index, kth coordinate system
$\bar{k}_a$	= unit vector along z axis of coordinate system a
n	= number of rotations between two unit vectors
N_a	= number of chains representing acceleration
N_n	= number of paths between two points in the Piogram
N_v	= number of chains representing velocity
${}'P^a_b$	= 4×4 D-H transformation matrix from coordinate system a to b about $\bar{j}_a$ ($\equiv \bar{j}_b$)
P^a_b	= coordinate transformation matrix from coordinate system a to b about $\bar{j}_a$ ($\equiv \bar{j}_b$)
P_k	= origin of kth coordinate system
${}'R^a_b$	= 4×4 D-H transformation matrix from coordinate system a to b about $\bar{i}_a$ ($\equiv \bar{i}_b$)
R^a_b	= coordinate transformation matrix from coordinate system a to b about $\bar{i}_a$ ($\equiv \bar{i}_b$)
$\bar{r}^a$	= position vector of point P_a with respect to the origin of system a
$\bar{r}^{ab}$	= sum of vector $\bar{r}^a$ and $\bar{r}^b$
r^{ab}_c or r^c_c	= column matrix whose entries are the $\bar{i}$, $\bar{j}$, and $\bar{k}$ components of $\bar{r}^{ab}$ or $\bar{r}^c$, coordinatized in system c
${}'r^{ab}_c$ or ${}'r^c_c$	= column matrix whose four entires are r^{ab}_c or r^c_c and 1
x_a	= x component of $\bar{r}^a$ in coordinate system a
x^{ab}_a	= x component of $\bar{r}^{ab}$ in coordinate system a
y_a	= y component of $\bar{r}^a$ in coordinate system a
y^{ab}_a	= y component of $\bar{r}^{ab}$ in coordinate system a
z_a	= z component of $\bar{r}^a$ in coordinate system a
z^{ab}_a	= z component of $\bar{r}^{ab}$ in coordinate system a
θ	= Euler angle of rotation about $\bar{j}_a$
ϕ	= Euler angle of rotation about $\bar{i}_a$
ψ	= Euler angle of rotation about $\bar{k}_a$
$(\dot{\ })$	= time derivative of $(\cdot)$
0	= origin of the initial (base) coordinate system

Received May 28, 1982; revision received Dec. 7, 1982.

*Associate Professor, Department of Aeronautical Engineering.

I. Introduction

A TYPICAL problem that one is faced with in the analysis and design of multidegree-of-freedom control systems, such as manipulators and robots, is that of computing the translatory motion of the edge of the last link. This involves coordinate transformations and computation of the position, velocity, and acceleration of the edge-point with respect to the manipulator base as functions of the values given to the independent variables which constitute the rotational and translational degrees of freedom. As an example of the complexity of the problem we turn to Fig. 1 which presents a combination of translatory and rotational degrees of freedom. Coordinate system 0 is attached to the base; vector $\bar{r}^0$ determines the location of point P_0. Prismatic (sliding) joints enable the three translatory degrees of freedom which determine the relative location of point P_0 with respect to point 0. The coordinates of P_0 in the 0 system are x_0, y_0, and z_0. Point P_0 is the origin of coordinate system 1 which is obtained when system 0 is rotated through the angle ψ about the $\bar{k}_0$ axis. This rotational degree of freedom is accorded to the system by a rotating joint at point P_0. This sequence of prismatic and rotating joints eventually leads to point P_3. (The termination of the sequence at point P_3 is, of course, arbitrary, and the ensuing developments will be valid for any length of sequence.)

We note that the rotations are Euler rotations; that is, each rotation is about an axis of the coordinate system which was obtained as a result of the rotation which immediately preceded the present one. There are several ways to express the relative orientation of one coordinate system with respect to another, e.g., Euler angles, direction cosine matrix, Euler vector, quaternions, etc. However, the most useful expression is the direction cosine matrix, since it enables us to easily transform vectors from one coordinate system to another. The classical way to obtain the direction cosine matrix (DCM) from the corresponding Euler angles is to write the DCMs that correspond to the single rotations and then multiply the string of DCMs. Pio[1,2] introduced a symbolic way to express coordinate transformations from which the DCM and the relation between the Euler angle rates and the angular velocities can easily be obtained. He has also shown how this symbolic representation, which is known as Piogram, can be used to analyze errors in Euler angle transformations.[3]

Experience has shown that after very little exercise one can easily master the usage of the Piogram, which then becomes a powerful analytic tool.

Denavit and Hartenberg[4] showed that the 3×3 DCM can be expanded into a 4×4 matrix, which not only accounts for rotations but also accounts for linear translations. This very useful matrix notation has been utilized thereafter in the computation of translatory position, velocity, and acceleration[5] and in other analyses related to multilink systems.[6]

While the DCM can be represented symbolically by the Piogram, to the best of our knowledge there is no similar symbolic representation of the 4×4 matrix of Denavit and Hartenberg. The purpose of this paper is to present a symbolic representation of the 4×4 matrix and its time derivatives, which are necessary for the computation of translatory positions, velocities, and accelerations.

In the next section, the Piogram and its rules will be presented. In Sec. III, the symbolic representation of the 4×4 matrix and its application to position computation will be introduced. Section IV will present the symbolic representation needed for velocity computation and Sec. V will present the same for acceleration computation. Section VI contains the conclusions.

In this paper the coordinate system locations and orientations, shown in Fig. 1, will serve as a baseline for the developments. The extension of this example to any number of links will be obvious. With no loss of generality, it will be assumed that all coordinate systems are right-hand systems and that all positive rotations are defined by the right-hand rule. In addition, if consecutive rotations are performed about the same axis, they are all lumped into one rotation.

II. Piograms

Consider the coordinate transformations shown in Fig. 1. We wish to compute the DCM which transforms vectors from system 3 to system 0. Denote this DCM by D_0^3. Obviously then

$$D_0^3 = A_0^1 P_1^2 R_2^3 \tag{1}$$

where

$$R_2^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \tag{2a}$$

$$P_1^2 = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{2b}$$

$$A_0^1 = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2c}$$

Using Eqs. (2) in Eq. (1) yields

$$D_0^3 = \begin{bmatrix} \cos\theta\cos\psi & \begin{matrix}\sin\phi\sin\theta\cos\psi \\ -\cos\phi\sin\psi\end{matrix} & \begin{matrix}\cos\phi\sin\theta\cos\psi \\ +\sin\phi\sin\psi\end{matrix} \\ \cos\theta\sin\psi & \begin{matrix}\sin\phi\sin\theta\sin\psi \\ +\cos\phi\cos\psi\end{matrix} & \begin{matrix}\cos\phi\sin\theta\sin\psi \\ -\sin\phi\cos\psi\end{matrix} \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix} \tag{3}$$

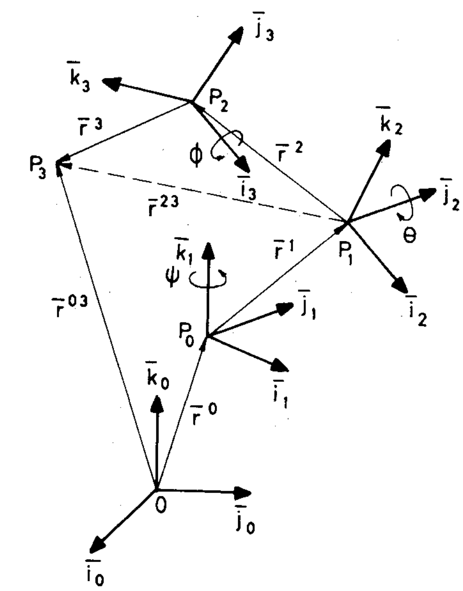

Fig. 1 Coordinate system location and attitude in a typical multilink mechanism.

Note that D_0^3 can be expressed as an inner product of the unit vectors along the axes of the 3 and the 0 coordinate systems as follows:

$$D_0^3 = \begin{bmatrix} \bar{i}_3\bar{i}_0 & \bar{j}_3\bar{i}_0 & \bar{k}_3\bar{i}_0 \\ \bar{i}_3\bar{j}_0 & \bar{j}_3\bar{j}_0 & \bar{k}_3\bar{j}_0 \\ \bar{i}_3\bar{k}_0 & \bar{j}_3\bar{k}_0 & \bar{k}_3\bar{k}_0 \end{bmatrix} \tag{4}$$

The expression for D_0^3 given in Eq. (3) can be obtained immediately, without matrix multiplications, using Piograms.[1,2] To meet this end, in Fig. 2 we draw the Piogram of the three rotations. In setting this Piogram, we use the following rules: 1) that a straight line is drawn between axes that stay identical in the rotation, and 2) that the angle by which the axes are rotated about a $\bar{j}$ axis is drawn in the corresponding circle with a reversed sign. To obtain the $\bar{i}_3\bar{i}_0$ element of D_0^3, we consider all paths starting at $\bar{i}_3$ and ending at $\bar{i}_0$. The number of paths is the number of terms in the $\bar{i}_3\bar{i}_0$ element. From Fig. 2 it is seen that there is only one such path. Now for this path we form a term which is the product of the sine and cosine of the angles, crossed by the path, using the following rules. Entrance and exit on the same side of the circle (i.e., upper or lower side) contributes to the product a cosine of the angle in the circle. Crossing a circle going up contributes a sine of the angle and going down contributes the negative of the sine of the angle. Accordingly, the single path from $\bar{i}_3$ to $\bar{i}_0$ results in the term

$$\bar{i}_3\bar{i}_0 = \cos\theta\cos(-\psi) = \cos\theta\cos\psi \tag{5}$$

Similarly the $\bar{j}_3\bar{i}_0$ term, which is the 1, 2 term of D_0^3, consists of two terms corresponding to two paths which, following the preceding rules, yield

$$\begin{aligned} \bar{j}_3\bar{i}_0 &= \cos(-\phi)\sin(-\psi) - \sin(-\phi)\sin\theta\cos(-\psi) \\ &= -\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi \end{aligned} \tag{6}$$

It can be seen that, using these rules, the rest of the terms in Eq. (3) can be easily obtained.

Pio[1,2] determined that the number of paths between two unit vectors (and thus the number of terms in the corresponding DCM element) is a Fibonacci number deter-

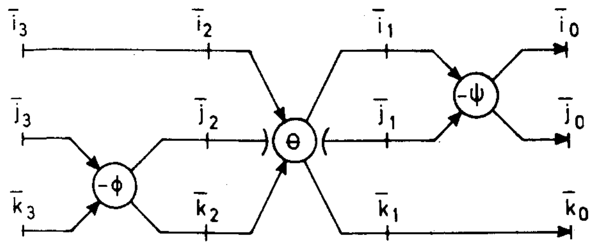

Fig. 2 Piogram of the coordinate transformations of Fig. 1.

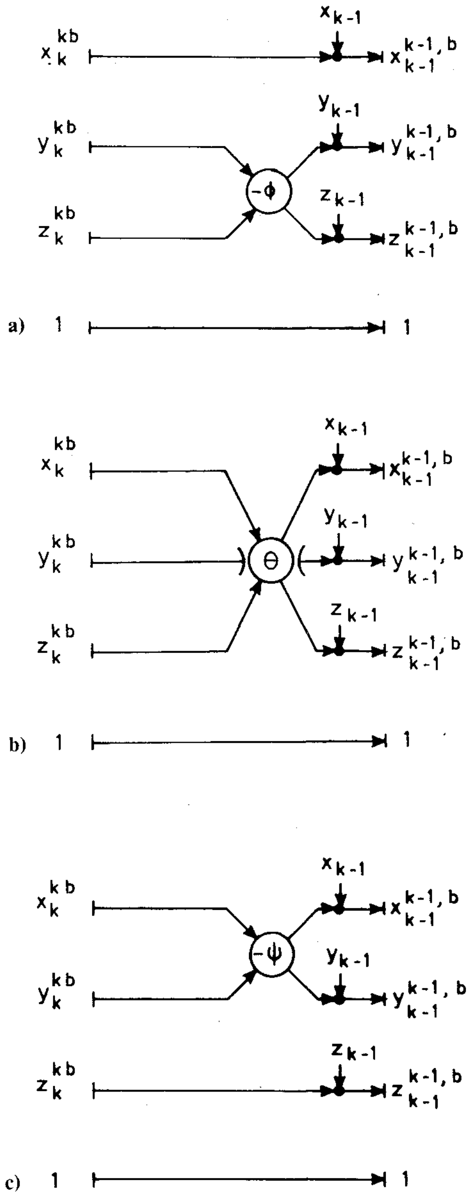

Fig. 3 Symbolic representation of all three D-H matrices: a) $|R_{k-1}^k$; b) $|P_{k-1}^k$; c) $|A_{k-1}^k$.

mined by the sequence

$$N_1 = N_2 = 1 \tag{7a}$$

$$N_k = N_{k-1} + N_{k-2} \tag{7b}$$

where N_n is the number of paths and n is the number of rotations between the two unit vectors. For example, it is seen in Fig. 1 that two rotations have placed $\bar{i}_0$ on $\bar{i}_3$. Thus $n=2$, and from Eq. (7a), $N_2 = 1$; hence the number of paths between $\bar{i}_0$ to $\bar{i}_3$ is 1. Similarly, it took three rotations to place $\bar{j}_3$ onto $\bar{j}_3$. Thus, here $n=3$ and, from Eq. (7b), $N_3 = N_2 + N_1$, which using Eq. (7a) yields $N_3 = 2$. The value of n can also be determined from the Piogram. It is the number of circle crossings obtained when moving along the path that crosses the maximal number of circles between the two unit vectors.

III. Symbolic Representation of Position Transformation

We turn again to Fig. 1. Suppose that R_2^3, the transformation matrix from coordinate system 3 to 2, is given, as well as the components of r_3^3 and r_2^2. It is desired to compute r_2^{23}. This can be easily done using

$$r_3^{23} = r_2^2 + R_2^3 r_3^3 \tag{8}$$

Denavit and Hartenberg[4] showed a different way of computing r_2^{23} by defining a 4×4 matrix as follows:

$$|R_2^3 = \left[\begin{array}{ccc|c} & & & x_2 \\ & R_2^3 & & y_2 \\ & & & z_2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{9}$$

Then using $|R_2^3$, r_2^{23} can be found as follows:

$$'r_2^{23} = |R_2^3\,'r_3^3 \tag{10}$$

The enormous advantage of the 4×4 Denavit and Hartenberg (D-H) matrix is that *consecutive* transformations, using the appropriate D-H matrices, yield the position vector of the edge-point P_3 with respect to point 0 resolved in the 0 (base) coordinate system. To demonstrate the above, define two more D-H matrices as follows:

$$|P_1^2 = \left[\begin{array}{ccc|c} & & & x_1 \\ & P_1^2 & & y_1 \\ & & & z_1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{11a}$$

$$|A_0^1 = \left[\begin{array}{ccc|c} & & & x_0 \\ & A_0^1 & & y_0 \\ & & & z_0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{11b}$$

Then

$$'r_3^{03} = |A_0^1\, |P_1^2\, |R_2^3\,'r_3^3 \tag{12}$$

When R_2^3, P_1^2, and A_0^1 given in Eqs. (2) are substituted into Eqs. (9), (11a), and (11b), respectively, and the results are substituted into Eq. (12), the following explicit relation is obtained:

$$\begin{bmatrix} x_0^{03} \\ y_0^{03} \\ z_0^{03} \\ 1 \end{bmatrix} = \begin{bmatrix} x_3\cos\theta\cos\psi + y_3(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi) \\ +z_3(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi) + x_2\cos\theta\cos\psi \\ -y_2\sin\psi + z_2\sin\theta\cos\psi - x_1\cos\psi - y_1\sin\psi + x_0 \\ x_3\cos\theta\sin\psi + y_3(\sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi) \\ +z_3(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi) + x_2\cos\theta\sin\psi \\ -y_2\cos\psi + z_2\sin\theta\sin\psi - x_1\sin\psi + y_1\cos\psi + y_0 \\ -x_3\sin\theta + y_3\sin\phi\cos\theta + z_3\cos\phi\cos\theta - x_2\sin\theta \\ +z_2\cos\theta + z_1 + z_0 \\ 1 \end{bmatrix} \tag{13}$$

Note that the D-H matrices are invertible; thus the inversion of $|R_2^3$, for example, yields

$$|R_3^2 = (\,|R_2^3)^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -x_2 \\ 0 & \cos\phi & \sin\phi & -(z_2\sin\phi + y_2\cos\phi) \\ 0 & -\sin\phi & \cos\phi & -z_2\cos\phi + y_2\sin\phi \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

and it can be easily seen that

$$'r_3^3 = (\,|R_2^3)^{-1}\,'r_2^{23} \tag{15}$$

which conforms with Eq. (10). Also note that, like the DCM, the determinant of the D-H matrix is equal to 1.

Unlike the DCM, which is represented symbolically by the Piogram, no symbolic representations have been given to date for the D-H 4×4 matrices. We propose the symbolic representations shown in Fig. 3. The four inputs to the symbolic representations are x_k^{kb}, y_k^{kb}, z_k^{kb}, and 1, respectively. According to our notations, the first three inputs are the components of r^{-kb}, which are obtained when r^{-kb} is coordinatized in system k. The outputs which are on the right of the symbolic representations are $x_{k-1}^{k-1,b}$, $y_{k-1}^{k-1,b}$, $z_{k-1}^{k-1,b}$, and 1. The first three output elements are the components of $r^{-k-1,b}$ when coordinatized in the $k-1$ system. The dots on the lines in the symbolic representations are summation points. To obtain the value of the outputs, one has to propagate all of the inputs as well as all of the vertical entries to each output through all possible paths using the Piogram rules. That is, each input and vertical entry has to be multiplied by the terms obtained when moving on all possible paths which start at that input or entry and end at the output under consideration. The expression for this term is obtained using the Piogram rules. Figure 4 illustrates, as an example, the outputs and inputs of a D-H $|R_{k-1}^k$ matrix.

The use of the D-H matrices to compute the components of $\bar{r}^{03}$ of Fig. 1 in coordinate system 0 was shown in Eq. (12). Figure 5 introduces the symbolic representation of the transformations of Eq. (12). Following the preceding rules, one can obtain the expressions given in Eq. (13). (Note that in constructing the symbolic representation of Eq. (12), moving from right to left along the transformations of Eq. (12), corresponds to moving from left to right in the symbolic representation of Fig. 5.) As an exercise, let us write the expression for z_0^{03}. Following the path from x_3 to z_0^{03} we compute the term $-\sin\theta$ which multiplies x_3 as well as x_2. Going from y_3, we obtain the term $\sin\phi\cos\theta$ which multiplies y_3, and going from z_3, the term is $\cos\phi\cos\theta$. Going from z_2 to z_0^{03}, we obtain $\cos\theta$, and going from z_1, we obtain 1. Note that no paths exist between other inputs or vertical entries and z_0^{03}. Consequently, z_0^{03} is constructed as follows:

$$z_0^{03} = -(x_3 + x_2)\sin\theta + y_3\sin\phi\cos\theta$$
$$+ z_3\cos\phi\cos\theta + z_2\cos\theta + z_1 + z_0 \tag{16}$$

This expression is identical to the one that was obtained for z_0^{03} in Eq. (13), where successive multiplications of the D-H matrices were used. The symbolic representation of Fig. 5 can, of course, be extended to any number of links using the basic blocks of Fig. 3.

Since the D-H matrices are invertible, it is obvious from Eq. (12) that

$$'r_3^3 = (\,|R_2^3)^{-1}(\,|P_1^2)^{-1}(\,|A_0^1)^{-1}\,'r_0^{03} \tag{17}$$

This result can also be expressed using the symbolic representations. To meet this end, simply reverse in Fig. 5 the direction of the signal flow such that the outputs on the right turn into inputs, whereas the inputs on the left turn into outputs. In addition, reverse the sign of the angles in the circles and subtract, rather than add, the vertical entries at the summation points. The symbolic representation of a single inverted D-H matrix is obvious.

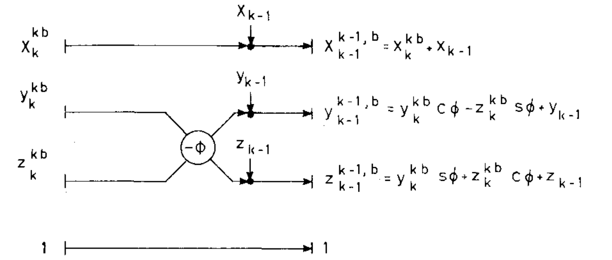

Fig. 4 Input/output relations for a D-H $|R_{k-1}^k$ matrix symbolic representation.

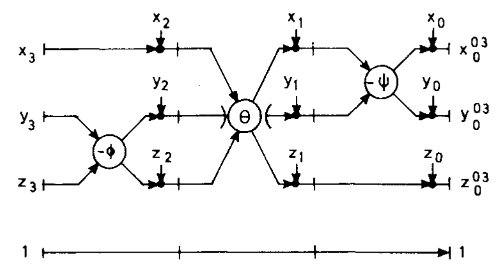

Fig. 5 Symbolic representation of the position transformation shown in Fig. 1.

IV. Symbolic Representation of Velocity Transformation

In order to obtain the velocity of the edge-point P_3 with respect to point 0, as seen by an observer at system 0, and coordinatized in this coordinate system, one may simply differentiate Eq. (13) with respect to time. This, however, can also be done using a symbolic representation similar to the one described in the preceding section. To meet this end, differentiate Eq. (12) to obtain

$$'\dot{r}_0^{03} = |\dot{A}_0^1\,|P_1^2\,|R_2^3\,'r_3^3 + |A_0^1\,|\dot{P}_1^2\,|R_2^3\,'r_3^3$$
$$+\ |A_0^1\,|P_1^2\,|\dot{R}_2^3\,'r_3^3 + |A_0^1\,|P_1^2\,|R_2^3\,'\dot{r}_3^3 \tag{18}$$

Using Eqs. (2), (9), and (11) it is found that

$$|\dot{A}_0^1 = \begin{bmatrix} -\sin\psi\dot{\psi} & -\cos\psi\dot{\psi} & 0 & \dot{x}_0 \\ \cos\psi\dot{\psi} & -\sin\psi\dot{\psi} & 0 & \dot{y}_0 \\ 0 & 0 & 0 & \dot{z}_0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{19a}$$

$$|\dot{P}_1^2 = \begin{bmatrix} -\sin\theta\dot{\theta} & 0 & \cos\theta\dot{\theta} & \dot{x}_1 \\ 0 & 0 & 0 & \dot{y}_1 \\ -\cos\theta\dot{\theta} & 0 & -\sin\theta\dot{\theta} & \dot{z}_1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{19b}$$

$$|\dot{R}_2^3 = \begin{bmatrix} 0 & 0 & 0 & \dot{x}_2 \\ 0 & -\sin\phi\dot{\phi} & -\cos\phi\dot{\phi} & \dot{y}_2 \\ 0 & \cos\phi\dot{\phi} & -\sin\phi\dot{\phi} & \dot{z}_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{19c}$$

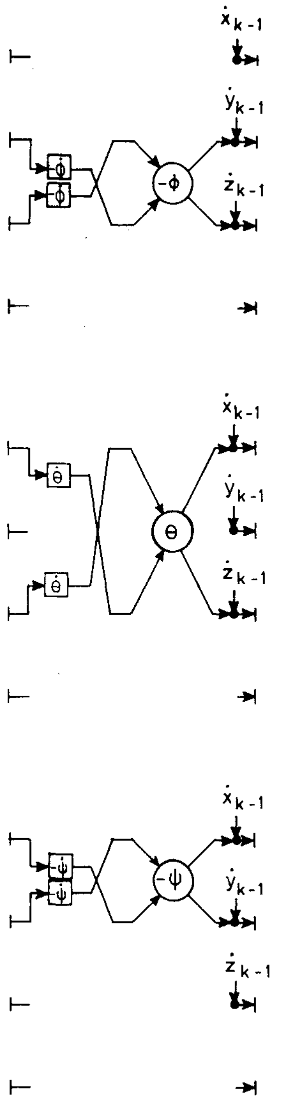

Fig. 6 Symbolic representation of the first time derivative of all three D-H matrices: a) $|\dot{R}^k_{k-1}$; b) $|\dot{P}^k_{k-1}$; c) $|\dot{A}^k_{k-1}$.

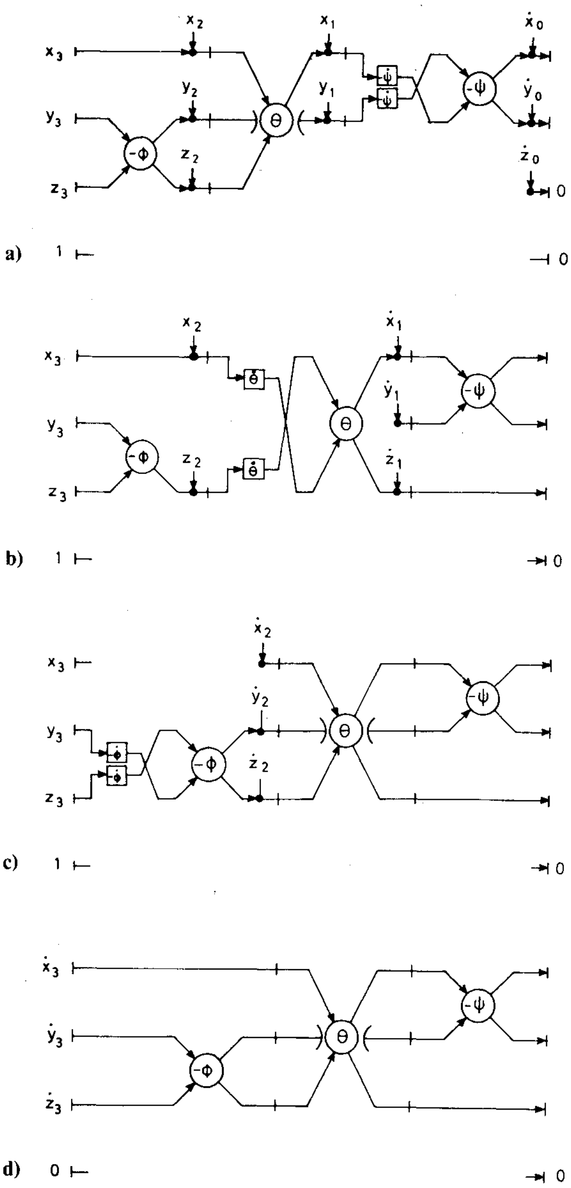

Fig. 7 Symbolic representation of the velocity transformation: a) $|\dot{A}^1_0\,|P^2_1\,|R^3_2{}'r^3_3$; b) $|A^1_0\,|\dot{P}^2_1\,|R^3_2{}'r^3_3$; c) $|A^1_0\,|P^2_1\,|\dot{R}^3_2{}'r^3_3$; d) $|A^1_0\,|P^2_1\,|R^3_2{}'\dot{r}^3_3$.

It can be seen that, using the preceding rules, the symbolic representations of $|\dot{A}^k_{k-1}$, $|\dot{P}^k_{k-1}$, and $|\dot{R}^k_{k-1}$ are those shown in Fig. 6 where the rectangles are gains whose value is enclosed in the rectangles. As in the Piograms, a path passing a rectangle going down reverses the sign of the term computed along the path.

Having found the symbolic representation of the first time derivative of all the D-H matrices, we can now draw in Fig. 7 the symbolic representations of all four terms in Eq. (18). As indicated by Eq. (18), the velocity vector of point P_3 with respect to point 0, when coordinated in system 0, is the sum of the corresponding outputs of the four diagrams in Fig. 7. Note an important rule in the symbolic representation of the velocity components; namely, going from input to output (left to right in our case) *there are no vertical inputs beyond the differentiated vertical input*, i.e., no vertical inputs beyond the symbolic representation of the differentiated D-H matrix.

It is obvious that N_v, the number of chains representing velocity, is given by

$$N_v = n + 1 \tag{20}$$

where n is the number of the D-H matrices in the expression for the position.

V. Symbolic Representation of Acceleration Transformation

To obtain the acceleration of point P_3 with respect to point 0 coordinatized in system 0, one can differentiate the expression for the velocity once or the expression for the position given in Eq. (13) twice. This, however, can also be accomplished directly using the symbolic representation. To show how this can be done, we differentiate Eq. (18) once with respect to time, yielding the following expression:

$$\begin{aligned}{}'\ddot{r}^{03}_0 &= |\ddot{A}^1_0\,|P^2_1\,|R^3_2{}'r^3_3 + |A^1_0\,|\ddot{P}^2_1\,|R^3_2{}'r^3_3 + |A^1_0\,|P^2_1\,|\ddot{R}^3_2{}'r^3_3 \\ &+ |A^1_0\,|P^2_1\,|R^3_2{}'\ddot{r}^3_3 + 2\,|\dot{A}^1_0\,|\dot{P}^2_1\,|R^3_2{}'r^3_3 + 2\,|A^1_0\,|\dot{P}^2_1\,|\dot{R}^3_2{}'r^3_3 \\ &+ 2\,|A^1_0\,|P^2_1\,|\dot{R}^3_2{}'\dot{r}^3_3 + 2\,|\dot{A}^1_0\,|P^2_1\,|\dot{R}^3_2{}'r^3_3 \\ &+ 2\,|A^1_0\,|\dot{P}^2_1\,|R^3_2{}'\dot{r}^3_3 + 2\,|\dot{A}^1_0\,|P^2_1\,|R^3_2{}'\dot{r}^3_3\end{aligned} \tag{21}$$

It can be shown that N_a, the number of terms in the acceleration expression, is given by

$$N_a = \tfrac{1}{2}(n^2 + 3n + 2) \tag{22}$$

where, as in Eq. (20), n is the number of the D-H matrices in the position expression. To find a symbolic representation of Eq. (21), we first have to know, symbolically, how to represent the second time derivative of the D-H matrices.

When we differentiate Eqs. (19), we obtain

$$|\ddot{A}_0^I = \left[\begin{array}{c|c|c|c} -\cos\psi\dot{\psi}^2 - \sin\psi\ddot{\psi} & \sin\psi\dot{\psi}^2 - \cos\psi\ddot{\psi} & 0 & \ddot{x}_0 \\ \hline -\sin\psi\dot{\psi}^2 + \cos\psi\ddot{\psi} & -\cos\psi\dot{\psi}^2 - \sin\psi\ddot{\psi} & 0 & \ddot{y}_0 \\ \hline 0 & 0 & 0 & \ddot{z}_0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \tag{23a}$$

$$|\ddot{P}_I^2 = \left[\begin{array}{c|c|c|c} -\cos\theta\dot{\theta}^2 - \sin\theta\ddot{\theta} & 0 & -\sin\theta\dot{\theta}^2 + \cos\theta\ddot{\theta} & \ddot{x}_I \\ \hline 0 & 0 & 0 & \ddot{y}_I \\ \hline +\sin\theta\dot{\theta}^2 - \cos\theta\ddot{\theta} & 0 & -\cos\theta\dot{\theta}^2 - \sin\theta\ddot{\theta} & \ddot{z}_I \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \tag{23b}$$

$$|\ddot{R}_2^3 = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & \ddot{x}_2 \\ \hline 0 & -\cos\phi\dot{\phi}^2 - \sin\phi\ddot{\phi} & \sin\phi\dot{\phi}^2 - \cos\phi\ddot{\phi} & \ddot{y}_2 \\ \hline 0 & -\sin\phi\dot{\phi}^2 + \cos\phi\ddot{\phi} & -\cos\phi\dot{\phi}^2 - \sin\phi\ddot{\phi} & \ddot{z}_2 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \tag{23c}$$

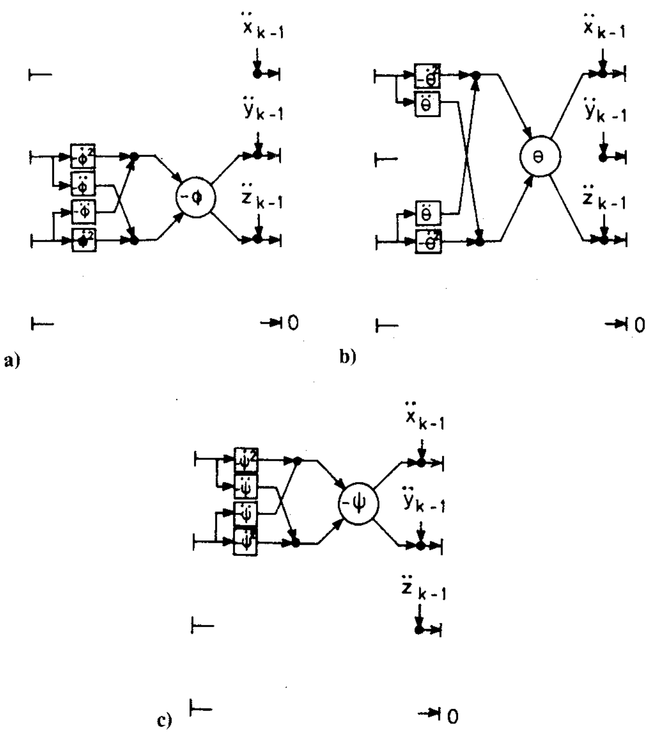

Fig. 8 Symbolic representation of the second time derivative of all three D-H matrices: a) $|\ddot{R}_{k-I}^k$; b) $|\ddot{P}_{k-I}^k$; c) $|\ddot{A}_{k-I}^k$.

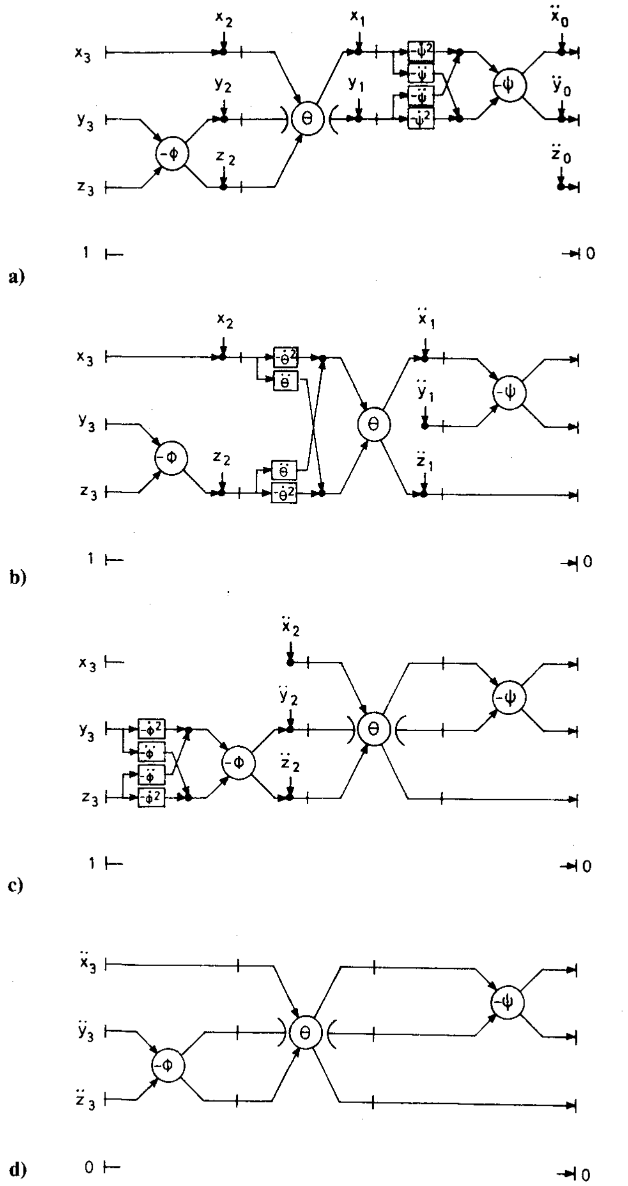

Fig. 9 Symbolic representation of the first four terms of the acceleration [Eq. (21)]: a) $|\ddot{A}_0^I\,|P_I^2\,|R_2^3{}'r_3^3$; b) $|A_0^I\,|\ddot{P}_I^2\,|R_2^3{}'r_3^3$; c) $|A_0^I\,|P_I^2\,|\ddot{R}_2^3{}'r_3^3$; d) $|A_0^I\,|P_I^2\,|R_2^3{}'\ddot{r}_3^3$.

The suitable symbolic representation of the matrices, given in Eqs. (23), is presented in Fig. 8. The preceding rules apply here also. It is easily seen that the symbolic representations shown in Fig. 8 do indeed represent the matrices of Eqs. (23). Having obtained the latter symbolic representations, we have all the necessary building blocks to construct the symbolic representation of Eq. (21). To accomplish this, we construct ten chains corresponding to the ten terms in Eq. (21). In drawing these chains we use the previously developed rules. In particular, note that any time derivative in the chain eliminates all the vertical entries in the symbolic representation of the following D-H matrices. (Remember, for this purpose, that moving from right to left in the analytic expressions corresponds to moving from left to right in the symbolic chains.) The components of the acceleration vector are finally obtained when the corresponding outputs of the ten chains are summed up after multiplying the appropriate terms by 2 as indicated in Eq. (21). For the sake of completeness, we draw in Figs. 9-11 all the ten chains corresponding to the ten terms of Eq. (21). The interested reader can prove for himself that each of these chains correctly represents the corresponding term in Eq. (21).

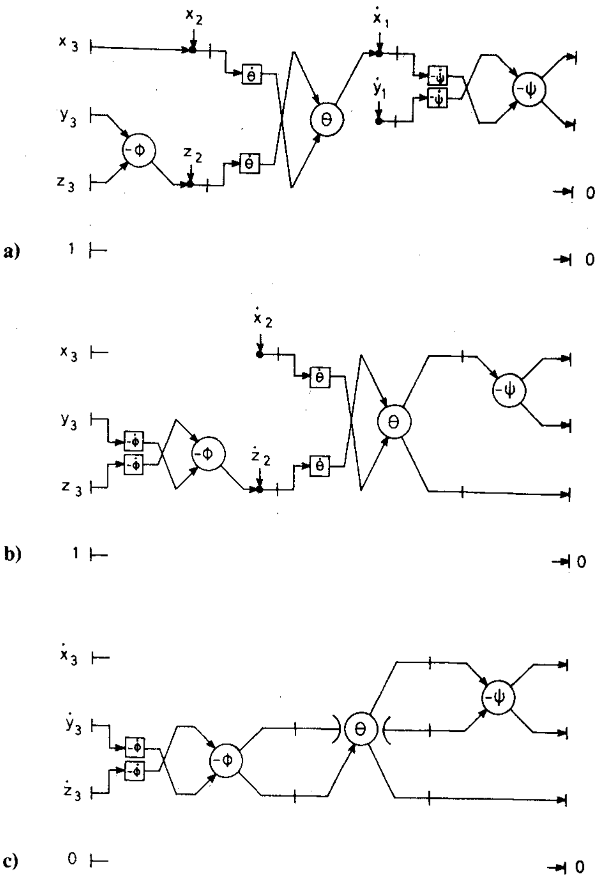

Fig. 10 Symbolic representation of the 5-7th terms of the acceleration [Eq. (21)]: a) $|\dot{A}_0^1|\dot{P}_1^2|R_2^3{}'r_3^3$; b) $|A_0^1|\dot{P}_1^2|\dot{R}_2^3{}'r_3^3$; c) $|A_0^1|P_1^2|\dot{R}_2^3{}'\dot{r}_3^3$.

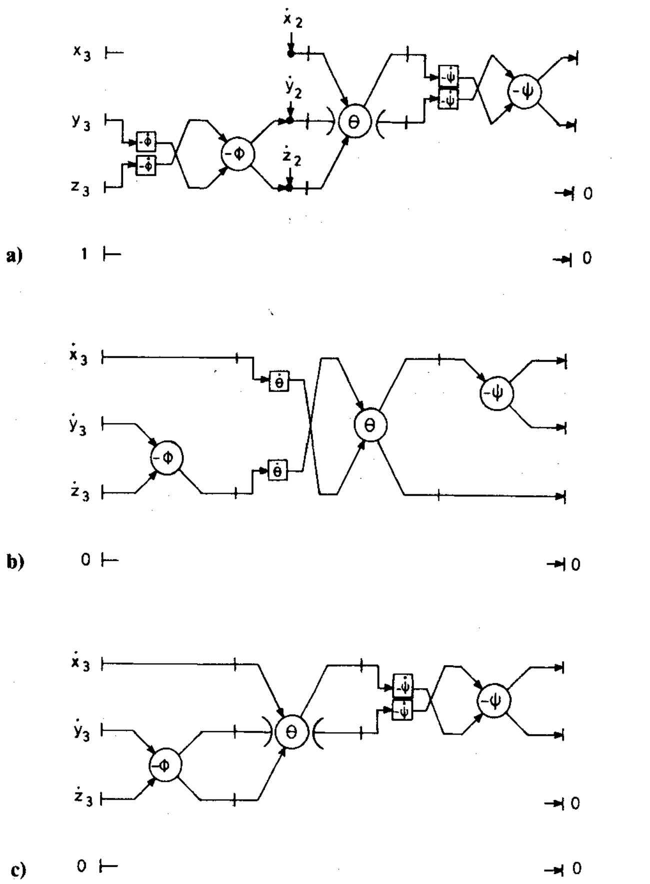

Fig. 11 Symbolic representation of the last three terms of the acceleration [Eq. (21)]: a) $|\dot{A}_0^1|P_1^2|\dot{R}_2^3{}'r_3^3$; b) $|A_0^1|\dot{P}_1^2|R_2^3{}'\dot{r}_3^3$; c) $|\dot{A}_0^1|P_1^2|R_2^3{}'\dot{r}_3^3$.

VI. Conclusions

Symbolic representations of position, velocity, and acceleration of the edge-point, with respect to the origin of a multivarying-link mechanism, have been presented. (The term "varying-link" means that the vectors, which determine the origin of the intermediate coordinate systems, may vary in length and in direction.) This symbolic representation is based on the successful symbolic representation of the D-H 4×4 matrices and their time derivatives, and on the Piogram. Although this work discussed, as an example, a chain of three transformations, the method can be extended to any number of successive rotations, since the number of the basic Euler rotations is only three and the symbolic representation of the corresponding three matrices and their two time derivatives have been presented here. The number of chains necessary to represent the velocity grows linearly with the number of rotations, and the number of chains necessary to represent the acceleration grows quadratically. This is unavoidable since the complexity of the problem and the number of terms in the analytic expressions have a similar growth. As with the use of Piograms, the biggest advantage in using the new symbolic representation is the elimination of the cumbersome matrix multiplications. Moreover, in the computation of velocity and acceleration using matrices, many terms are computed which eventually are dropped (since at the end of the chain of matrix multiplications they are multiplied by zero). This unnecessary computation is avoided using the symbolic representation, because it is clearly seen that the path which involves these terms quickly reaches a dead end. It was found that, as with Piograms, once the user masters the new technique, it becomes a very easy and powerful tool for obtaining the analytic expressions for position, velocity, and acceleration in multilink mechanisms. Finally, the new symbolic representation can be coded into an efficient computer program, which can generate the analytic expression for the components of the translatory motion of any link in the mechanism.

Acknowledgment

The author wishes to thank Moshe Shoham of the Mechanical Engineering Department for his useful discussions and for reviewing this work.

References

[1] Pio, R.L., "Symbolic Representation of Coordinate Transformation," *IEEE Transactions on Aerospace and Navigational Electronics,* Vol. ANE-11, June 1964, pp. 128-134.

[2] Pio, R.L., "Euler Angle Transformation," *IEEE Transactions on Automatic Control,* Vol. AC-11, Oct. 1966, pp. 707-715.

[3] Pio, R.L., "Error Analysis of Euler Angle Transformation," *AIAA Journal,* Vol. 11, June 1973, pp. 871-875.

[4] Denavit, J. and Hartenberg, R.S., "A Kinematic Notation for Lower Pair Mechanisms Based on Matrices," *Transactions of ASME, Journal of Applied Mechanics,* Vol. 22, June 1955, pp. 215-221.

[5] Denavit, J., Hartenberg, R.S., Razi, R., and Vicker, J.J., "Velocity, Acceleration, and Static-Force Analyses of Spatial Linkages," *ASME Journal of Applied Mechanics,* Vol. 32, Dec. 1965, pp. 903-910.

[6] Sunada, W. and Dubowsky, S., "The Application of Finite Element Methods to the Dynamic Analysis of Flexible Spatial and Co-Planar Linkage Systems," *ASME Journal of Mechanical Design,* Vol. 103, July 1981, pp. 643-651.